Date: 01-04-2020
Lecture - 03

Vector (Unit - 04)

|sem → Civil + EC| (10)
Subject: - Basic Math
Code: - 160113/1602203

* PRODUCT OF TWO VECTORS (DOT PRODUCT)

(i) Scalar (or Dot) product :- Scalar product of two vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle b/w $\vec{a}$ and $\vec{b}$ $(0 \leq \theta \leq \pi)$.

(ii) From definition we derive :-

(a) $\bar{a}\cdot\bar{b}$ is a scalar quantity.

(b) when $\theta = 0$, $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|$

Also $\bar{a}\cdot\bar{a} = |\bar{a}|^2 = |\bar{a}||\bar{a}|$

(c) when $\theta = \pi/2$, $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\pi/2 = 0$

$\Rightarrow$ when $\bar{a} \perp \bar{b}$, $\bar{a}\cdot\bar{b} = 0$

(d) when either $\vec{a} = 0$ or $\vec{b} = \vec{0}$, $\vec{a}\cdot\vec{b} = 0$

(e) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\bar{a}||\bar{b}|}$

(f) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{i} = 1.1.\cos 0 = 1$
$\hat{j}\cdot\hat{j} = 1.1.\cos 0 = 1$
$\hat{k}\cdot\hat{k} = 1.1.\cos 0 = 1$

(g) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i}$

$\hat{i}\cdot\hat{j} = 1.1.\cos 90^\circ = 0$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

$|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\bar{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$\therefore \cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$

(iii) Properties of Scalar product:-

(a) $\bar{a}\cdot\bar{b}$ is commutative i.e.

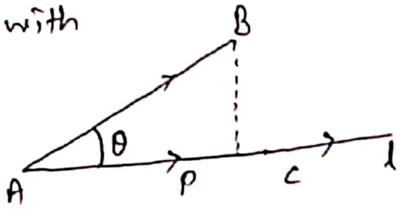
$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

(b) If α is scalar, then

$$(\alpha\bar{a})\cdot\bar{b} = \alpha(\bar{a}\cdot\bar{b}) = \bar{a}\cdot(\alpha\bar{b})$$

* Projection of a vector along a directed line:-

Let the vector $\bar{AB}$ makes an angle θ with directed line l.



Projected of AB on l.

$$= |\bar{AB}|\cos\theta = |\bar{AC}| = \bar{P}$$

The vector $\bar{p}$ is called the projection vector.
Its magnitude is $|\bar{P}|$. which is known as projection of vector $\bar{AB}$.
The angle θ b/w $\bar{AB}$ and $\bar{AC}$ is given by

$$\cos\theta = \frac{\bar{AB}\cdot\bar{AC}}{|\bar{AB}||\bar{AC}|}$$

Now projection $AC = |\bar{AB}|\cos\theta = \frac{|\bar{AB}||\bar{AC}|\cos\theta}{|\bar{AC}|} = \frac{\bar{AB}\cdot\bar{AC}}{|\bar{AC}|}$

$$= \bar{AB}\cdot\left(\frac{\bar{AC}}{|\bar{AC}|}\right)$$

If $\bar{AB} = \bar{a}$ $\quad \bar{AC} = \bar{P}$ $\quad \therefore AC = \bar{a}\cdot\left(\frac{\bar{P}}{|\bar{P}|}\right) = \bar{a}\cdot\hat{p}$

Thus the projection of $\bar{a}$ on $\bar{b}$

$$= \bar{a}\cdot\left(\frac{\bar{b}}{|\bar{b}|}\right) = \bar{a}\cdot\hat{b}$$

Note:- If α, β, γ are the direction angles of the vector $\bar{a} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$, the direction cosines of $\bar{a}$ are given as

$$\cos\alpha = \frac{\bar{a}\cdot\hat{i}}{|\bar{a}||\hat{i}|} = \frac{a_1}{|\bar{a}|}, \quad \cos\beta = \frac{a_2}{|\bar{a}|} \quad \cos\gamma = \frac{a_3}{|\bar{a}|}$$

Ex!. Find the angle b/w two vectors $\bar{a}$ and $\bar{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\bar{a}\cdot\bar{b} = \sqrt{6}$.

Solution!. We know that the angle θ b/w two vectors $\bar{a}$ & $\bar{b}$ is given by

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$\because |\bar{a}| = \sqrt{3}$

$|\bar{b}| = 2$

$\bar{a}\cdot\bar{b} = \sqrt{6}$

$$= \frac{\sqrt{6}}{(\sqrt{3})(2)} = \frac{(\sqrt{3})(\sqrt{2})}{\sqrt{3}(2)}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$= \cos\pi/4$$

$$\theta = \pi/4 \quad \text{Ans.}$$

Ex!. Evaluate the product $(3\bar{a} - 5\bar{b})\cdot(2\bar{a} + 7\bar{b})$

Solution!. $(3\bar{a} - 5\bar{b})\cdot(2\bar{a} + 7\bar{b})$

$$= 3\bar{a}\cdot(2\bar{a} + 7\bar{b}) - 5\bar{b}(2\bar{a} + 7\bar{b})$$

$$= 6\bar{a}\cdot\bar{a} + 21\bar{a}\cdot\bar{b} - 10\bar{b}\cdot\bar{a} - 35\bar{b}\cdot\bar{b}$$

$$= 6|\bar{a}|^2 + 21\bar{a}\cdot\bar{b} - 10\bar{b}\cdot\bar{a} - 35|\bar{b}|^2 \qquad (\because \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a})$$

$$= 6|\bar{a}|^2 + 11\bar{a}\cdot\bar{b} - 35|\bar{b}|^2 \quad \text{Ans.}$$

Ex!. Find the projection of the vector $i + 3j + 7k$ on the vector $7i - j + 8k$.

Solution!. Let $\bar{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ $\quad \bar{b} = 7i - j + 8k$

Then projection of $\bar{a}$ on $\bar{b}$ $= \dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$ ——— (1)

$\bar{a}\cdot\bar{b} = (\hat{i} + 3\hat{j} + 7\hat{k})\cdot(7i - \hat{j} + 8\hat{k}) = (1)(7) + (3)(-1) + (7)(8)$

$= 7 - 3 + 56 = 60$

$|\bar{b}| = \sqrt{(7)^2 + (-1)^2 + (8)^2} = \sqrt{49 + 1 + 64}$

$= \sqrt{114}$

From (1), we have

projection of $\bar{a}$ on $\bar{b}$ $= \dfrac{60}{\sqrt{114}}$ Ans.

Ex.: Find the angle b/w the vectors:-

$$(2\hat{i}+6\hat{j}+3\hat{k}) \text{ and } (12\hat{i}-4\hat{j}+3\hat{k})$$

Soln.: We know that the angle θ b/w two vectors $\bar{a}$ & $\bar{b}$ is given by

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$$

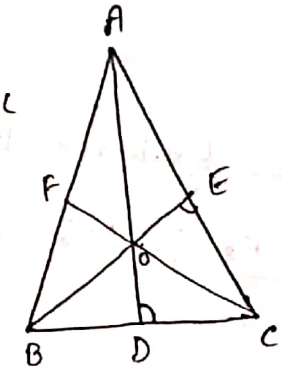
$$= \frac{(2\hat{i}+6\hat{j}+3\hat{k})\cdot(12\hat{i}-4\hat{j}+3\hat{k})}{\left(\sqrt{(2)^2+(6)^2+(3)^2}\right)\left(\sqrt{(12)^2+(-4)^2+(3)^2}\right)}$$

$$= \frac{24-24+9}{7\cdot 13} = \frac{9}{91}$$

$$\theta = \cos^{-1}\frac{9}{91} \quad \text{Ans.}$$

Ex.: Prove that the altitudes of a triangle are concurrent.

Soln.: Let the altitudes AD & BE meet at O.
Let $\bar{a}, \bar{b}$ & $\bar{c}$ are position vectors of A, B, C w.r.t O as origin. Join CO.



Since $\vec{AO} \perp \vec{BC}$

$$\bar{AO}\cdot\bar{BC} = 0$$
$$-\bar{a}\cdot(\bar{c}-\bar{b}) = 0$$
$$-\bar{a}\cdot\bar{c}+\bar{a}\cdot\bar{b} = 0 \quad \text{—(i)}$$

Since $\bar{BO} \perp \bar{AC}$

$$\bar{BO}\cdot\bar{AC} = 0$$
$$-\bar{b}\cdot(\bar{c}-\bar{a}) = 0$$
$$-\bar{b}\cdot\bar{c}+\bar{b}\cdot\bar{a} = 0 \quad \text{—(ii)}$$

On subtracting (i) from (ii), we have

$$-\bar{b}\cdot\bar{c}+\bar{a}\cdot\bar{c} = 0$$
$$-\bar{c}\cdot(\bar{b}-\bar{a}) = 0$$
$$\vec{CO}\cdot\vec{AB} = 0$$

$\therefore$ $\vec{CO}$ is perpendicular to $\vec{AB}$.
Hence, the three altitudes meet at a point.

Exercise

1. Find the value of $(2\hat{i}+3\hat{j}+4\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})$ Ans $= 9$

2. Find the projection of vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$ Ans $= 0$

3. If $\bar{a}, \bar{b}, \bar{c}$ are unit vector such that $\bar{a}+\bar{b}+\bar{c}=0$, then find the value of $\bar{a}\cdot\bar{b}+\bar{b}\cdot\bar{c}+\bar{c}\cdot\bar{a}$ Ans $= -\frac{3}{2}$

4. Find the angle b/w the vectors:-

(i) $\bar{a} = 3\hat{i}+\hat{j}+2\vec{k}$ $\quad \bar{b} = 2\hat{i}-2\hat{j}+4\hat{k}$ Ans - $\cos^{-1}\sqrt{3/7}$

(ii) $\bar{a} = \hat{i}-\hat{j}$, $\quad \bar{b} = \hat{j}+\hat{k}$ Ans $= 120°$

5. Find $\bar{a}\cdot\bar{b}$, if

(i) $\bar{a} = 3\hat{i}-4\hat{j}+7\hat{k}$ $\quad \bar{b} = 2\hat{i}-\hat{j}$ Ans :- $= 10$

(ii) $\bar{a} = \hat{i}-\hat{j}+5\hat{k}$ $\quad \bar{b} = 2\hat{i}+3\hat{j}-\hat{k}$ Ans $= -6$

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