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Date :- 02-04-2020
Period :- 1st (10:30 - 11:30 AM)

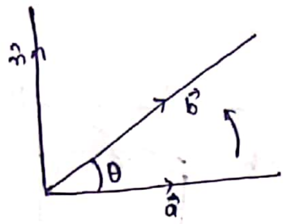
Vector (Unit - 04) Lecture - 04

Sem.- Civil + EC
Subject :- Basic Maths
Code :- 1601103/1602203

* Vector product of two Vectors :-

(1) Definition :- The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$ is defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta . \hat{n}$$



where θ is the angle b/w $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$. unit vector $\hat{n}$ is perpendicular to both vectors $\vec{a}$ and $\vec{b}$, such that $\vec{a}.\vec{b}$ and $\hat{n}$ from a right handed system.

* NOTE :-

(a) $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta . \hat{n}$

(i) If $\vec{a} = 0$ or $\vec{b} = 0$, $\vec{a} \times \vec{b} = 0$

(ii) $\vec{a} \parallel \vec{b}$, $\vec{a} \times \vec{b} = 0$

(b) $\vec{a} \times \vec{b}$ is a vector, i.e. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\therefore \vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

$\Rightarrow \vec{a} \times \vec{b}$ is not commutative.

(c) when $\theta = \pi/2$, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \times \hat{n}$

or $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$

(d) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

$\Rightarrow \hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$

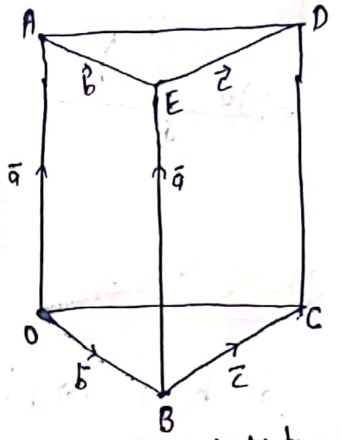
(e) $$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

(f) If $\vec{a}$ and $\vec{b}$ represent adjacent sides of a parallelogram then its area $|\vec{a} \times \vec{b}|$.

(g) If $\vec{a}, \vec{b}$ represent the adjacent sides of a triangle, then its area $= \frac{1}{2}|\vec{a} \times \vec{B}|$.

(h)* **DISTRIBUTIVE LAW:-**

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$



Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors

$\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{BC} = \vec{c}$

and $\vec{BE} = \vec{a}$, $\vec{AE} = \vec{b}$, $\vec{ED} = \vec{c}$

Complete the parallelograms OAEB, BCDE and OADC.

R.H.S $\rightarrow$ $\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ = Vector Area of parallelogram OAEB + Vector Area of para-llelogram ~~parallelogram~~ BCDE

= Vector Area of OAEDCB

= Vector Area of OAEDCB − Vector area of ΔOCB + Vector area of ΔADE ($\because \Delta OBC = \Delta ADE$)

= vector area of ||gm OADC.

$= \vec{OA} \times \vec{AD}$

$= \vec{a} \times (\vec{b} + \vec{C})$ L.H.S.

so $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(i) Let α be a scalar, then

$$\alpha(\vec{a} \times \vec{b}) = (\alpha\vec{a}) \times \vec{b} = \vec{a} \times (\alpha\vec{b})$$

(j)* **Vector product expressed as a determinant:-**

If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Solved Examples

Ex:-1 → If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $(\vec{a}\times\vec{b}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution:- given $|\vec{a}| = 2$; $|\vec{b}| = 7$ and $\vec{a}\times\vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$$\therefore \vec{a}\times\vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(3)^2+(2)^2+(6)^2} = \sqrt{9+4+36} = \sqrt{49} = 7$$

we know that the angle θ b/w two vectors $\vec{a}$ & $\vec{b}$ is given by

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\cancel{7}}{2\times\cancel{7}} = \frac{1}{2} = \cos\pi/6$$

$$\therefore \theta = \pi/6 \quad \text{Ans.}$$

Ex:-2. If $a = 3\hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} + \hat{j} + 5\hat{k}$ then find the angle b/w $(\vec{a}\times\vec{b})$ and $(\vec{a}+\vec{b})$. (2019 EVEN)

Solution:-
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ 4 & 1 & 5 \end{vmatrix} = \hat{i}(10-1) - \hat{j}(15-4) + \hat{k}(3-8)$$
$$= 9\hat{i} - 11\hat{j} - 5\hat{k}$$

$$\vec{a}+\vec{b} = (3\hat{i}+2\hat{j}+\hat{k}) + (4\hat{i}+\hat{j}+5\hat{k}) = 7\hat{i}+3\hat{j}+6\hat{k}$$

$$(\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}) = (9\hat{i}-11\hat{j}-5\hat{k})\cdot(7\hat{i}+3\hat{j}+6\hat{k})$$
$$= 63 - 33 - 30 = 0$$

$$|\vec{a}\times\vec{b}| = |9\hat{i} - 11\hat{j} - 5\hat{k}| = \sqrt{(9)^2+(-11)^2+(-5)^2}$$
$$= \sqrt{81+121+25} = \sqrt{227}$$

$$|\vec{a}+\vec{b}| = |7\hat{i}+3\hat{j}+6\hat{k}| = \sqrt{(7)^2+(3)^2+(6)^2} = \sqrt{49+9+36} = \sqrt{94}$$

$$\therefore \cos\theta = \frac{(\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b})}{|\vec{a}\times\vec{b}|\,|\vec{a}+\vec{b}|} = \frac{0}{(\sqrt{227})(\sqrt{94})} = 0 = \cos\pi/2$$

$$\theta = \pi/2 \quad \text{Ans.}$$

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Example! (3) If $\vec{a} = 5\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{c} = 3\vec{i} + 2\vec{j} + 7\vec{k}$ then calculate $(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})$

(Diploma 2019 EVEN)

Solution!. $\vec{a} = 5\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}$, $c = 3\vec{i} + 2\vec{j} + 7\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} = \vec{i}(2-9) - \vec{j}(5-6) + \vec{k}(15-4)$$
$$= -7\vec{i} + \vec{j} + 11\vec{k}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 3 & 2 & 7 \end{vmatrix} = \vec{i}(21-2) - \vec{j}(14-3) + \vec{k}(4-9)$$
$$= 19\vec{i} - 11\vec{j} - 5\vec{k}$$

$$(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c}) = (-7\vec{i} + \vec{j} + 11\vec{k})\cdot(19\vec{i} - 11\vec{j} - 5\vec{k})$$

$$= -199 \text{ Ans.}$$

Example! (4) Prove that $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = 0$

(Diploma 2019 ODD)

$\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$

$L.H.S = \vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$

$= (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a}) + (\vec{c}\times\vec{a}) + (\vec{c}\times\vec{b})$

$= (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c}) - (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c}) - (\vec{b}\times\vec{c})$

$\boxed{\because\ \vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})}$

$= 0$ R.H.S.

Example! (5) Find the unit vector perpendicular to each of the vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $-3\hat{i} - 2\hat{j} + \hat{k}$.

(December 2004)

Solution!. $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -3\hat{i} - 2\hat{j} + \hat{k}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = 8\hat{i} - 10\hat{j} + 4\hat{k}$$

$\vec{a}\times\vec{b}$ is a vector perpendicular to $\bar{a}$ and $\bar{b}$.

Unit perpendicular vector $= \dfrac{8\hat{i} - 10\hat{j} + 4\hat{k}}{\sqrt{(8)^2 + (-10)^2 + (4)^2}} = \dfrac{1}{\sqrt{180}}(8\hat{i} - 10\hat{j} + 4\hat{k})$ Ans

Exercise

1. Find $\vec{a} \times \vec{b}$ if $a = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ Ans = $\hat{i} + 2\hat{j} - \hat{k}$

2. If $\vec{a} = 3\hat{i} - 2\hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, calculate $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$.
   Answer = $-8\hat{i} + 14\hat{j} + 26\hat{k}$

3. Find a unit vector perpendicular to both of the vectors $A = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + 2\hat{k}$
   Ans:- $\dfrac{\hat{i} - 5\hat{j} - 3\hat{k}}{\sqrt{35}}$

4. Find a unit vector perpendicular to the plane of ~~the~~ vector $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$
   Ans:- $\dfrac{\hat{i} - 5\hat{j} - 3\hat{k}}{\sqrt{35}}$

5. Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if $|\vec{a} \times \vec{b}| = \vec{a} \cdot \vec{b}$
   Ans:- $\pi/4$

6. If $\vec{a} + \vec{b} + \vec{c} = 0$, prove that $(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a})$

7. If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq 0$, then show that $\vec{a} + \vec{c} = k\vec{b}$, where k is a scalar.

8. Find a vector of magnitude $\sqrt{171}$ which is perpendicular to both of the vectors $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$.
   Answer = $\pm(\hat{i} - 11\hat{j} - 7\hat{k})$

9. $\vec{a} = \hat{i} - \hat{j} - 3\hat{k}$, $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, and $\vec{c} = 2\hat{i} - \hat{j} + 2\hat{k}$, verify that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$